BEAM EXTRACTION NEAR A THIRD-INTEGRAL RESONANCE II

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In this report, we will obtain Eq. (I-1)* from the primitive equations of motion for betatron oscillations, and in particular, we will find expressions for the variables γ, ρ and the parameter A in terms of the natural variables x, $x' = dx/d\Theta$ and the accelerator parameters. We take for x the radial displacement from the equilibrium orbit, and $\Theta = s/R$, where s is the distance along the equilibrium orbit measured from the extraction septum, and $2\pi R$ is the length of the equilibrium orbit.

The linear equation of motion

$$x'' + g(\Theta)x = 0 \tag{1}$$

can be deduced from the Hamiltonian

$$h_2 = \frac{1}{2}x'^2 + \frac{1}{2}g(\Theta)x^2 \tag{2}$$

where $x' = dx/d\Theta$ is the momentum conjugate to x. With sextupole, octupole,...terms present, the Hamiltonian is

$$h = h_2 + h_3(x, x', \Theta) + h_4(x, x', \Theta) + \cdots, \tag{3}$$

where h_3, h_4 are cubic and quartic in x, x', and periodic in Θ.

The solution of the linearized equation can be expressed in terms of the Floquet solution

*Equations from report I of the series under this title will be numbered in this fashion in this report. Equation numbers without Roman prefixes refer to the present report.

$$x = we^{i(\nu\Theta + \psi - \pi/2)} \tag{4}$$

and its complex conjugate, where $w(\Theta)$, $\psi(\Theta)$ are periodic with period 2π if misalignments are included, otherwise with period $2\pi/N_s$ where $N_s = 6$ is the number of superperiods. Choosing an appropriate linear combination of x and x^*, we may write the general solution in the form

$$x = Aw\sin(\nu\Theta + \psi + \zeta), \tag{5}$$

$$x' = Aw'\sin(\nu\Theta + \psi + \zeta) + Aw^{-1}\cos(\nu\Theta + \psi + \zeta),$$

where A, ζ are arbitrary amplitude and phase. The matrix $M(\Theta)$ which carries x, x' from Θ to $\Theta + 2\pi$ is, from Eq. (5),

$$\mathbf{M}(\Theta) = \begin{pmatrix} \cos 2\pi\nu - \alpha\sin 2\pi\nu & \beta\sin 2\pi\nu \\ -\frac{1+\alpha^2}{\beta}\sin 2\pi\nu & \mathrm{Cos}\, 2\pi\nu + \alpha\sin 2\pi\nu \end{pmatrix}, \tag{6}$$

where $\alpha = ww'$, $\beta = w^2 = (\nu + \psi')^{-1}$. (7)

It is convenient to choose the additive constant in ψ so that $\psi\ (\Theta = 0) = 0$:

$$\psi(\Theta) = \int_0^{\Theta} (\beta^{-1} - \nu)d\Theta. \tag{8}$$

We now can write the solution (5) in the form

$$x = Pw\sin\psi + Xw\cos\psi, \tag{9}$$

$$x' = P(w'\sin\psi + w^{-1}\cos\psi) + X(w'\cos\psi - w^{-1}\sin\psi),$$

where

$$X = A\sin(\nu\Theta + \zeta), \tag{10}$$

$$P = A\cos(\nu\Theta + \zeta).$$

We can readily verify that the Poisson bracket

$$\frac{\partial x}{\partial X}\frac{\partial x'}{\partial P} - \frac{\partial x}{\partial P}\frac{\partial x'}{\partial X} = 1, \tag{11}$$

so that X, P are also canonical variables which clearly satisfy equations derived from the Hamiltonian

$$H_2 = \frac{1}{2}\nu(X^2 + P^2). \tag{12}$$

It is easily verified that the transformation (8) indeed leads to the Hamiltonian (12). We note that at $\theta = 0$,

$$\begin{aligned} x &= w_0 X = \beta_0{}^{1/2} X, \\ x' &= w_0{}^{-1} P + w' X = \beta_0{}^{-1/2}(P + \alpha X), \\ X &= \beta_0{}^{-1/2} x, \\ P &= \beta_0{}^{1/2} x' - \alpha\beta_0{}^{-1/2} x. \end{aligned} \tag{13}$$

Since the transformation (9) is canonical in any case, and since its generating function is quadratic, we may carry out the same transformation on the nonlinear Hamiltonian (3) merely by replacing h_2 by H_2 and making the substitution (9) in h_3, h_4,...

We now suppose that there is a distribution of sextupoles around the machine azimuth, giving rise to fields

$$\begin{aligned} B_z &= -F(\theta)(x^2 - z^2), \\ B_x &= -2F(\theta)xz^2. \end{aligned} \tag{14}$$

Here z is the vertical coordinate, and $F(\theta)$ is the sextupole strength. The contributions of the sextupole terms to the equations of motion are given by

$$\begin{aligned} x'' &= \text{linear terms} + \frac{eRF(\theta)}{M\gamma\omega}(x^2 - z^2), \\ z'' &= \text{linear terms} - \frac{2eRF(\theta)}{M\gamma\omega}xz. \end{aligned} \tag{15}$$

We are primarily interested in the radial motion, but we also need to insure that the z-motion is not unduly disturbed by the extraction scheme. The cubic terms in the Hamiltonian which give the sextupole terms in Eqs. (18) are

$$h_3 = \frac{eRF(\Theta)}{M\gamma\omega}\,(xz^2 - 1/3x^3). \tag{16}$$

The linear z-motion can be treated in precise analogy with the treatment above of the x-motion. We are led to a cononical transformation $z,\ z' \rightarrow Z,\ P_z$ analogous to the transformation (9). We will ignore for the moment the z-term in h_3, as the other term is the one which drives the extraction resonance.

Before calculating H_3, we note that we may introduce canonical polar coordinates ρ, γ in the X, P-plane via the transformation

$$\begin{aligned} X &= (2\rho)^{1/2}\sin\gamma, \\ P &= (2\rho)^{1/2}\cos\gamma. \end{aligned} \tag{17}$$

It is readily verified that the necessary Poisson bracket condition is satisfied. We will transform between X, P and γ, ρ whenever convenient. In particular, we have from Eq. (9):

$$x = (2\rho\beta)^{1/2}\sin\,(\gamma + \psi), \tag{18}$$

so that the amplitude of x-oscillations is given by $(2\rho\beta)^{1/2}$.

We now make the substitution (9) or (18) in the x-term in Eq. (16):

$$H_3 = \frac{eRF(\Theta)}{12M\gamma\omega}\,(2\rho\beta)^{3/2}[\sin\,(3\gamma + 3\psi) - 3\,\sin\,(\gamma + \psi)]. \tag{19}$$

We assume that the sextupoles are located at azimuths Θ_j, $j = 1$, 2,..., and put

$$F(\Theta) = \sum_j F_j \delta(\Theta - \Theta_j). \tag{20}$$

We Fourier analyze H_3:

$$H_3 = \frac{eR}{24\pi M\gamma\omega} \sum_{jm} F_j e^{im(\Theta - \Theta_j)} (2\rho\beta_j)^{3/2} [\sin(3\gamma + 3\psi_j) - 3\sin(\gamma + \psi_j)]$$

$$= (2\rho)^{3/2} \sum_{m=-\infty}^{\infty} \left[H_{33m} \cos(3\gamma - m\Theta + \eta_{33m}) + H_{31m} \cos(\gamma - m\Theta + \eta_{31m}) \right], \tag{21}$$

where

$$H_{33m} e^{i\eta_{33m}} = \frac{eR}{24\pi M\gamma\omega} \sum_j \beta_j^{3/2} F_j e^{i(m\Theta_j + 3\psi_j - \pi/2)} \tag{22}$$

$$H_{31m} e^{i\eta_{31m}} = \frac{eR}{8\pi M\gamma\omega} \sum_j \beta_j^{3/2} F_j e^{i(m\Theta_j + \psi_j + \pi/2)}$$

It can be seen from Eq. (8) that the ψ_j are equal at homologous points around the accelerator. In particular, if the azimuth of sextupole j is homologous to the extraction septum, $\psi_j = 0$.

The resonance $\nu = m_0/3$ is driven by the term

$$H_{33m_0} (2\rho)^{3/2} \cos(3\gamma - m_0\Theta + \eta_{33m_0}). \tag{23}$$

If ν is very close to $^{m}0/3$, this will be the dominant sextupole term. All other terms can be transformed into higher orders in ρ by a suitable change of variables. In a later report we will carry out this transformation in order to determine the resulting fourth order terms in H_4 which distort the separatrices found in report I. The resulting variables after the transformation are only slightly different from X, P or ρ, γ defined above, provided no other nearby resonance is strongly driven by a term in Eq. (21). The term in H_{31m} can drive the integral resonance $\nu = m$; hence it is desirable to have several sextupoles at carefully chosen azimuths Θ_j and with appropriate amplitudes F_j so as to make H_{31m} vanish for at least the nearest integer m to ν. If the Θ_j are all homologous, so that the β_j and ψ_j are all equal, then H_{33m} (and H_{31m} also) vanishes if $F(\Theta)$ has no mth Fourier component. This can be arranged for n-1 components if there are n sextupoles with suitably chosen F_j, Θ_j. However the requirement that they be at homologous points may necessitate up to 2n sextupoles, to eliminate n-1 harmonics and provide a desired amplitude and phase of the harmonic m_0. Although the terms in Eq. (21) cannot drive a half-integral resonance, any deviation of the equilibrium orbit from the center of the sextupoles will introduce quadrupole terms which can. The terms driving the resonance $\nu = m/2$ can be eliminated by eliminating the mth harmonic from $F(\Theta)$ provided that the Θ_j are homologous and the orbit deviations are identical in each. Otherwise the elimination of H_{22m} is more complicated and may depend on the orbit deviation.

The term xz^2 in Eq. (16) drives resonances of the type $\nu_x \pm 2\nu_z = m$. We can readily calculate the amplitude $H_{1\pm 2m}$ of the corresponding driving terms in the same way as above. It will probably be desirable so to place the sextupoles as also to eliminate or minimize the terms that drive the one or two sum and difference resonances closest to the working point.

It should be noted that one advantage of working at a third-integral resonance is that this resonance selects out the quadratic terms in the equations of motion, the term (23) having a dominant effect, while terms which drive other resonances have relatively less effect. If we try to extract on an integral resonance, on the other hand, depending nevertheless on sextupole terms to increase the amplitude at the extraction point, then not only sextupole terms, but also gradient terms and field bumps can affect the orbit strongly. The resonance $\nu = m$ is also $\nu = 2m/2$ and $\nu = 3m/3$. If the equilibrium orbit is centered in the sextupoles, then the sextupole terms, which drive $\nu = 3m/3$ may be dominant. However, any deviation of the orbit from the center of the sextupoles will introduce gradient and field bumps which drive $\nu = 2m/2$ and $\nu = m$. Thus the integral extraction process can be extremely sensitive to orbit deviations.

We now write the approximate Hamiltonian for ν near the third integral resonance by adding the term (23) to the quadratic Hamiltonian H_2 given by Eq. (12):

$$H = \nu\rho + H_{33m_0}(2\rho)^{3/2} \cos\,(3\gamma - m_0\Theta + \eta_{33m_0}). \tag{24}$$

We introduce a final transformation $\rho,\gamma \to \underline{\rho},\underline{\gamma}$ via the generating function

$$\begin{aligned} S &= \underline{\rho}(\gamma - 1/3m_0\Theta), \\ \rho &= \partial S/\partial\gamma = \underline{\rho}, \\ \underline{\gamma} &= \partial S/\partial\underline{\rho} = \gamma - 1/3m_0\Theta. \end{aligned} \tag{25}$$

The new Hamiltonian is

$$\begin{aligned} \underline{H} &= H + \partial S/\partial\Theta \\ &= (\nu - m_0/3)\underline{\rho} + H_{33m_0}(2\underline{\rho})^{3/2} \cos\,(3\underline{\gamma} + \eta_{33m_0}). \end{aligned} \tag{26}$$

This is the form introduced in report I, where we dropped the bar from $\underline{\rho}$. The coefficient $A = H_{33m_0}$ is given by Eq. (22)

and the azimuths Θ_j and sextupole strengths F_j must be chosen so that $\eta_{33m_0} = 0$.

Note that the barred phase plane $\underline{\rho}$, $\underline{\gamma}$, or $\underline{X}$, $\underline{P}$, rotates with angular velocity $m_0/3$ relative to the X, P-plane, returning to its original position every three revolutions. Since the curves in Fig. I-1 have a three-fold symmetry, the phase plot shown in that figure repeats every revolution.